J. Fluid Mech. (1976), *vol.* 76, *part* 3, *pp.* 501–517
Printed in Great Britain

Stress relaxation effect in elastico-viscous lubricants in gears and rollers

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(Received 16 September 1974 and in revised form 12 March 1976)

The stress relaxation effect in an incompressible elastico-viscous fluid is investigated for the case of the lubrication of line-contact rollers. Two extreme cases are considered: that of low pressures with rigid surfaces and constant viscosity and that of heavily loaded elasto-hydrodynamic lubrication. In order to solve this problem, a new invariant material time derivative is suggested. This derivative is referred to a co-ordinate system attached to the principal axes of the strain-rate tensor, while the former derivatives have been referred to the fluid particle. It is shown that, unlike the previous derivatives, the new one enables a separate parametric description of the stress relaxation process and the first normal-stress difference. The results show that a significant increase in the load capacity is obtained, owing to the relaxation time of the fluid. The investigation is for fluids with a relaxation time small compared with the transit time of the lubricant through the bearing.

1. Introduction

The aim of this study is to solve the problem of the hydrodynamic rolling bearing using an elastico-viscous fluid model in which the stress relaxation effect is of first order compared with the zero-order Newtonian viscosity. The relaxation process is described by the fluid time τ (also called the relaxation time). The magnitude of this elastico-viscous parameter can be determined experimentally from the phase lag between the stress and strain rate in oscillatory flow.

Hydrodynamic lubrication of gear-teeth and rolling-element bearings involves rapid changes in the shear rates of the fluid particles. Accordingly Cameron (1953), in view of many experiments, suggested that in such cases the relaxation time of ordinary mineral oils may have a significant role in improving the performance of the lubricant. Many lubrication fluids in common use today contain additives consisting of high molecular weight polymers. The addition of polymers to mineral oils makes them much more elastico-viscous in the sense that their relaxation time becomes much longer than that for ordinary mineral oils. It is well known that, unlike Newtonian fluids or ordinary mineral oils, elastico-viscous lubricants (such as polymer solutions) exhibit shear-rate and temperature dependent viscosity and normal stresses in shear flow. In order to isolate these complex rheological properties, only the stress relaxation effect is investigated here. It has been shown by Pearson (1967) that in one interpretation of hydrodynamic lubrication the non-Newtonian viscosity is the dominant zero-order effect, while the contributions of the other elastico-viscous effects are of first or higher order.

2. Relation to previous investigations

The present study is a continuation of many previous investigations carried out in an effort to find the role of elastico-viscous fluids in lubrication. There is still disagreement between the experimental and theoretical results. Several experimental studies have suggested that elastico-viscous lubricants are superior in performance, but there is not yet any satisfactory theoretical explanation; for details, see recent discussions by Appeldorn (1968), Walters (1972) and Harnoy (1974). Most of the previous analytical work dealt with hydrodynamic journal bearings and led to the following conclusions (Reiner, Hanin & Harnoy 1969; Harnoy 1971; Davies & Walters 1972; Harnoy & Hanin 1974).

(i) The occurrence of a negative first normal-stress difference in simple shear of elastico-viscous liquids increases the load capacity of the lubrication film. In a simple shear flow, $u = \dot{\gamma}y$ and $v = w = 0$ (u, v and w are the velocity components in the orthogonal Cartesian directions x, y and z respectively). The first normal-stress difference $S_{xx}-S_{yy}$ is between the direction of shear and the normal to it. The ratio of $S_{xx}-S_{yy}$ to the shear stress S_{xy} (this ratio is called the Weissenberg number) must reach the order of magnitude of 10^4 in order to cause practical improvement capable of explaining the experiments with steady-state ordinary bearings.

(ii) The second normal-stress difference $S_{yy}-S_{zz}$ does not affect the load capacity in a full film, infinitely long bearing. In short bearings the effect is negligible.

(iii) The improvement cannot be accounted for by the different curve of viscosity *vs.* shear rate and temperature; see Horowitz & Steidler (1960).

These analytical results imply that the Weissenberg number must be very high. However, this assumption is very much in doubt and is still a matter of controversy. There is no experimental evidence for such a high first normal-stress difference and the maximum experimental values of the Weissenberg number reported for lubricants are of the order of thirty. Philippoff (1968) compared the results of different experimental methods. Since the previous analytical investigations are still in disagreement with the experiments, more research on the role of elastico-viscous fluids in lubrication is needed.

The phenomenon of hydrodynamic lubrication, as well as the rheology of elastico-viscous lubricants at high shear rates, involves a large number of parameters, which complicate experimental investigation and its interpretation. In order to understand better the performance of elastico-viscous lubricants, a separate investigation of each parameter and its effect in hydrodynamic lubrication would be advantageous. For any analytical treatment, approximate constitutive equations accounting for the effect of each parameter separately are needed.

3. Relaxation effect in lubrication

The present study suggests that the stress relaxation effect is a separable property of elastico-viscous fluids at high shear rates. It is shown that this effect leads to a possible explanation of the experimental results. When a fluid particle flows along the lubrication film, it is subject to changing shear stresses and shear rates. The phase lag between the shear stress and shear rate must affect the stress and pressure distribution along the lubrication film. Separate investigation of this effect is very important in order to decide whether it improves or degrades the lubrication. A demonstration of improvement of lubrication would mean that the assumption of a very high first normal-stress difference would not necessarily be the only one available to explain the experimental results. In fact it is found that the stress relaxation process does improve the load-carrying capacity of the fluid film in gears and rollers. The effect would then be a Deborah-number rather than a Weissenberg-number effect.

Milne (1957) and Burton (1960) recognized first the importance of investigating separately the effect of stress relaxation in lubrication films. Their studies contributed to the advance of the research in this subject, but after the progress in rheology in recent years, some of their assumptions must be re-examined, especially the constitutive equation, since it is not frame invariant and the velocity distribution was not derived but assumed.

The reason why the role of relaxation in lubrication has not yet been thoroughly investigated is because available material time derivatives, or any combination of them, made it impossible to separate the first normal-stress difference and the relaxation process.

4. Stress relaxation and first normal-stress difference

The stress relaxation and $S_{xx} - S_{yy}$ are not always independent variables for any type of flow. It is well known that in slow flow the two effects are governed by the same fundamental parameter. Coleman & Noll (1960) have shown that the second-order equation of Rivlin & Ericksen (1955) represents the first perturbation from Newtonian fluids for slow flows. This second-order equation describes the first normal-stress difference and the stress relaxation by the same parameter. The connexion between the two effects, in slow flow, is confirmed by molecular or suspension models. A Newtonian fluid particle in simple shear is subject to tension and compression in the directions 45° from the shear plane. At the same time it is rotating and changing its angular position with respect to the principal stress components. This is the reason why in elastico-viscous fluids 'memory' effects account for the first normal-stress difference as well as for the phase lag in oscillatory flow.

In contrast, it will be shown that at high shear rates, in order to describe the real behaviour of elastico-viscous fluids, the two effects must be described by two distinct parameters capable of separate experimental determination. The fluid time parameter which governs the slow changes cannot any longer describe the real normal stresses resulting from the high-speed angular rotation.

A careful check on the contribution of each effect in lubrication has shown that any one fluid time parameter which described correctly the relaxation effect would predict unreasonably high $S_{xx}-S_{yy}$ values. On the other hand, a parameter that described the measured $S_{xx}-S_{yy}$ would practically neglect the relaxation effect. Most of the earlier theoretical investigations of hydrodynamic lubrication with elastico-viscous fluids assumed the second-order-fluid equation for the lubricant. The significance of these analyses has been in detecting the effects of the normal stresses in lubrication. Apparently, a separate investigation is needed to find the role of the stress relaxation process.

5. Requirements for constitutive equation

For separate investigation of the stress relaxation effect, it is convenient to formulate a semi-empirical constitutive equation that will meet the following requirements.

(*a*) The constitutive equation must show a phase lag between the stress and rate of strain in oscillatory flow.

(*b*) The equation must result in no normal stresses in simple shear flow.

(*c*) The relation between the rate of strain and stress in the material must be unaffected by the choice of the frame of reference. The response of the material must not be changed by an arbitrary rigid rotation or translation of the reference co-ordinates. This means that the properties of the material are independent of the observer and should not be affected by the particular co-ordinate system, at rest or in motion, which happens to be employed. (This requirement in continuum mechanics is divided into two principles: co-ordinate invariance and material objectivity.)

(*d*) The equation must show isotropy in the rest state because the fluid has no preferred directions before its deformation.

In order to describe the real physical situation at high shear rates, semi-empirical constitutive equations can be obtained only by modifying the usual definitions of the time derivative of the strain-rate tensor. The second-order equation, with this new definition for the material time derivative, is found to be suitable for the present problem of flow at high shear rates. At the same time the new equation complies with the requirements of continuum mechanics.

6. Second-order-fluid equation and material time derivatives

The second-order-fluid equation of Rivlin & Ericksen (1955) is

$$S_{ij} = -P\delta_{ij} + 2\alpha_1 e_{ij} + 2\alpha_2 De_{ij}/Dt + 4\alpha_3 e_{i\alpha} e_{\alpha j} \tag{1}$$

(for simplicity, the co-ordinate system is rectangular Cartesian, which is suitable for describing the flow in lubrication problems). Here P is the hydrostatic pressure and S_{ij} and e_{ij} are respectively the stress and the rate-of-strain tensor, where

$$e_{ij} = \tfrac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i), \tag{2}$$

in which the v_i are the velocity components and the x_i the reference co-ordinates. $\alpha_1 = \eta$ is the fluid viscosity and α_2 and α_3 are second-order coefficients of an elastico-viscous fluid. There are various definitions for the material time derivative D/Dt, as discussed by Prager (1961). The material time derivatives in common use are the covariant and contravariant derivatives of Oldroyd, the strain rates of Rivlin & Ericksen and the co-rotational derivative of Jaumann, which reads as follows:

$$\frac{D}{Dt}e_{ij} = \frac{\partial}{\partial t}e_{ij} + \frac{\partial e_{ij}}{\partial x_\alpha}v_\alpha - \omega_{i\alpha}e_{\alpha j} + e_{i\alpha}\omega_{\alpha j}, \tag{3}$$

where the ω_{ij} are the angular-velocity components of a fluid particle:

$$\omega_{ij} = \tfrac{1}{2}(\partial v_i/\partial x_j - \partial v_j/\partial x_i). \tag{4}$$

The material time derivatives have to be invariant under rigid rotation and translation of the frame of reference, with respect to the continuum. The rule of transformation for a tensor $\mathbf{T}$ in one frame of reference to $\mathbf{T}'$ in another, relatively rotating, frame is

$$\mathbf{T}' = \mathbf{RTR}^*, \tag{5}$$

where $\mathbf{R}$ is the rotation matrix and $\mathbf{R}^*$ is its transpose. For the co-rotational derivative, the same rate-of-strain tensor relative to the material is obtained whether e_{ij} is in steady or rotating co-ordinates.

It is easy to see that in a rigid co-ordinate system (x_1, y_1, z_1) attached to a fluid particle, rotating and moving with it, (3) reduces to a partial time derivative. In (x_1, y_1, z_1) co-ordinates we get

$$\omega_{ij} = 0, \quad v_i = 0 \tag{6}$$

and

$$\frac{D}{Dt}e_{ij} = \frac{\partial}{\partial t}e_{ij}(x_1; y_1; z_1). \tag{7}$$

The Jaumann derivative [equation (3)] can be derived by transforming the partial derivative in (x_1, y_1, z_1) to any arbitrary frame of reference x_i. The Jaumann derivative is invariant because it is defined in the invariant co-ordinates (x_1, y_1, z_1), attached to a fluid particle and independent of the frame of reference.

In a simple steady shear flow,

$$u = \dot{\gamma}y, \quad v = w = 0, \tag{8}$$

though the flow is steady in time and space, the Jaumann derivative of the strain rate is not zero. The reason is the rotation of the fluid particles relative to the rate-of-strain tensor. The rotation rate around the z axis is

$$\omega = \tfrac{1}{2}\dot{\gamma}. \tag{9}$$

For a second-order fluid in simple shear, the Weissenberg number is

$$(S_{xx} - S_{yy})/S_{xy} = -2\dot{\gamma}\alpha_2/\alpha_1. \tag{10}$$

It is important to emphasize that a careful check showed the same result for all the available time derivatives, or any combination of them. $-\alpha_2/\alpha_1$ has the dimensions of time and depends only on the fluid. This ratio is called the fluid time or relaxation time, and will be denoted by

$$\tau = -\alpha_2/\alpha_1. \tag{11}$$

τ is always positive, becau e α_2 is always negative. Substituting (9) and (11) in (10) yields

$$(S_{xx}-S_{yy})/S_{xy} = \tau\omega. \tag{12}$$

The previous solutions for lubrication problems involving a second-order fluid showed that the pressure distribution and the load capacity are governed by one parameter: the ratio $-\alpha_2/\alpha_1$ (in addition to the viscosity). According to (10) this ratio describes the role of the first normal-stress difference along the sheared lubrication film.

With respect to co-ordinates attached to a fluid particle, the lubrication flow is unsteady. A fluid particle is subjected to alternating shear rates while passing along the bearing. In a second-order fluid, the relaxation effect in an unsteady flow is determined by the same ratio $-\alpha_2/\alpha_1 = \tau$. One can demonstrate this effect by the phase lag for a periodic shear rate

$$e_{xy} = \dot{\gamma}_0 \cos \omega_1 t. \tag{13}$$

Substituting (13) in the second-order equation (1), the shear stress is

$$S_{xy} = |S_{xy}| \cos(\omega_1 t - \psi), \tag{14}$$

where
$$\tan\psi = (-\alpha_2/\alpha_1)\,\omega_1 = \tau\omega_1. \tag{15}$$

Comparison of (12) and (15) indicates that the second-order equation is valid only if measurements in steady simple shear and oscillatory shear have the same ratio τ. Phase-lag measurements for practical elastico-viscous lubricants give fluid times higher than $O(10^{-4})$ s. The shear rate in lubrication reaches $O(10^7)\ \mathrm{s}^{-1}$, thus the ratio $(S_{xx}-S_{yy})/S_{xy}$ according to (10) is $O(10^3)$. In contrast, the maximum measured value for this ratio is less than 100. These orders of magnitude show that one coefficient $(-\alpha_2/\alpha_1 = \tau)$ cannot predict the exact orders of magnitude of the two effects at high shear rates.

Equations (12) and (15) show that in a second-order fluid the first normal-stress difference is basically the same 'memory effect' as the relaxation, both being involved with a fluid time parameter. The reason why the fluid time parameters are not equal at the high shear rates of lubrication is that the frequency of rotation of the particles is several orders of magnitude higher than the frequency of periodic changes in a fluid particle passing along the bearing:

$$\omega/\omega_1 = O(10^3\text{–}10^4). \tag{16}$$

It may be expected that the elastico-viscous molecular mechanism involved at low frequencies is different from that at high frequencies of rotation. The time required for the fluid to pass along the bearing is $O(l/U)$, where l is the length of the lubrication slit and U the circumferential velocity of the bearing. The periodic time of the fluid particle rotation is $O(h/U)$, where h is the film thickness. The ratio of these two times is $O(l/h)$, therefore the ratio of the frequencies is $O(\omega/\omega_1) = O(l/h)$. The film thickness in lubrication is very thin, $O(l/h) = 10^3\text{–}10^4$, which leads to the ratio given in (16).

Metzner, White & Denn (1966) showed that the second-order approximation for a fluid with a 'memory' is valid only when the relaxation time τ is small compared with the characteristic time of the flow Δt. When the flow is periodic, Δt is of the order of magnitude of the period $[\Delta t = O(1/\omega)]$. The ratio $\tau/\Delta t$ is called the Deborah number N_{Deb} (see Reiner 1964):

$$N_{\mathrm{Deb}} = O((-\alpha_2/\alpha_1)\,\omega_1) = O(\tau\omega_1). \tag{17}$$

For liquids with fluid times $O(10^{-4})$ s and $\omega_1 = O(10^3)$, the Deborah number with respect to the time required for a fluid particle to pass along the bearing is

$$\tau\omega_1 = O(10^{-1}). \tag{18}$$

With respect to the rotation time of a fluid particle, another Deborah number can be considered:

$$\tau\omega = O(10^3). \tag{19}$$

Equation (18) shows that the relaxation effect can be described by the second-order equation, since the Deborah number involved is small relative to unity. The first normal-stress difference, which is due to the high-speed rotation of the particles, must be described by other means. The reason is the relatively high Deborah number involved in this effect, according to (19). (Nevertheless, in simple steady shear, the second-order equation describes the real behaviour of elastico-viscous fluids at high shear rates also, because there is only one frequency of variation of the rate of strain with respect to the rotating fluid particles and a single suitable fluid time parameter can be chosen in this case.)

7. Separation of the relaxation and the first normal-stress difference

In order to separate these two effects, the material time derivative has to be defined in other invariant reference co-ordinates. The reference co-ordinates suggested here are the principal co-ordinates of the derived tensor. The time derivative is defined in a rigid rectangular co-ordinate system $(1, 2, 3)$. *The origin of* $(1, 2, 3)$ *is attached to the fluid particle and moves with it, but its directions always coincide with the three principal axes of the rate-of-strain tensor*. The definition of the suggested new tensor d/dt is a partial time derivative in the system $(1, 2, 3)$ attached to the principal axes of the derived tensor:

$$\frac{d}{dt}e_{ij} = \frac{\partial}{\partial t}e_{ij}(1, 2, 3). \tag{20}$$

By definition, d/dt describes the rate of change of the principal components of the derived tensor, being unaffected by the rotation of the principal directions. d/dt is independent of the choice of the arbitrary frame of reference x_i because the directions of the principal axes $(1, 2, 3)$ are invariants of this tensor.

By transforming the derivative d/dt to an arbitrary frame of reference x_i, according to (5), one gets, in a similar way to the Jaumann derivative,

$$\frac{d}{dt}e_{ij} = \frac{\partial}{\partial t}e_{ij} + \frac{\partial e_{ij}}{\partial x_\alpha}v_\alpha - \Omega_{i\alpha}e_{\alpha j} + e_{i\alpha}\Omega_{\alpha j}, \tag{21}$$

where the Ω_{ij} are the angular-velocity components of the rigid co-ordinate system $(1, 2, 3)$ relative to x_i (around the axis x_k, $k \neq i, j$). A formal proof that

(21) is invariant under arbitrary rotation of the reference co-ordinates x_i is given in the appendix. According to (21), the same tensor d/dt is obtained for e_{ij} in either steady or rotating co-ordinates.

Unlike the Jaumann derivative, the principal directions of the tensors e_{ij} and de_{ij}/dt coincide. The principal components of d/dt are the rates of change of the principal components $e_{\alpha\alpha}$ in the same directions.

The usefulness of the new time derivative can be demonstrated by its capability of describing the relaxation process alone. In contrast to all the previous material time derivatives, in steady shear flow [equation (8)] $de_{ij}/dt = 0$, because the principal axes of e_{ij} do not rotate ($\Omega = 0$). When d/dt is employed in the second-order-fluid equation we get

$$(S_{xx} - S_{yy})/S_{xy} = 0. \tag{22}$$

On the other hand, for a periodic shear rate, the same phase lag according to (15) is obtained with the new time derivative d/dt as with the previous derivatives. These results show how d/dt permits separation of the two effects.

Resolving the angular velocity ω_{ij} around the axis $k \neq i,j$, we have

$$\omega_{ij} = \Omega_{ij} + \overline{\omega}_{ij}, \tag{23}$$

where ω_{ij} is the angular velocity of a fluid particle relative to an arbitrary frame of reference x_i, Ω_{ij} is that of the rigid system $(1, 2, 3)$ relative to x_i and $\overline{\omega}$ is that of the particle relative to $(1, 2, 3)$. It is important to emphasize that $\overline{\omega}$ is independent of the rotation of x_i.

d/dt differs from D/Dt by the terms $-\overline{\omega}_{i\alpha} e_{\alpha j} + e_{i\alpha}\overline{\omega}_{\alpha j}$. Since $\overline{\omega}_{ij}$ and e_{ij} are independent of the frame of reference x_i, these terms are also invariant, and represent the rate of change of e_{ij} due to the rotation $\overline{\omega}$.

Introducing an additional parameter into the second-order fluid, an invariant constitutive equation is obtained as follows:

$$S_{ij} = -P\delta_{ij} + 2\alpha_1 e_{ij} + 2\alpha_2 de_{ij}/dt + 2\alpha_3(-\overline{\omega}_{i\alpha} e_{\alpha j} + e_{i\alpha}\overline{\omega}_{\alpha j}) + 4\alpha_4 e_{i\alpha} e_{\alpha j}, \tag{24}$$

where α_2 is the parameter of relaxation, obtainable from experiments with alternating rates of strain. The terms involving α_3 and α_4 describe the first and second cross-stress differences. For slow flow $\alpha_2 = \alpha_3$ and (24) reduces to the regular second-order equation. $-\alpha_3/\alpha_1$ is the fluid time for high frequency rotation, while $-\alpha_2/\alpha_1$ is the relaxation time for the lower frequency linear changes.

Pearson (1967) showed that, in the equations of flow for thin-film lubrication, the elastico-viscous terms are of first and higher order in perturbation expansions about the zero-order terms in Reynolds' equation for non-Newtonian viscous fluid. It is possible by this perturbation method to find approximate solutions for the contribution of every first-order term separately. Thus, in (24), by retaining only one of the three terms containing α_2, α_3 and α_4, it is possible to get separate perturbation solutions for the relaxation and first and second normal-stress differences respectively (their mutual effect being of higher order and negligible).

Moreover, if the experimental values for the relaxation time and cross-stresses are accepted, then in lubrication flow the orders of magnitude of the two normal-stress terms are lower than that of the relaxation term. Comparison of the previous analytical results with those of the present study indicates that for $\alpha_2 = \alpha_3$ the relaxation and the first normal-stress difference are of the same order in the differential equations. (α_4 has no effect on the load capacity.) For the measured values of the relaxation time, greater than 10^{-4} s, and for a Weissenberg number $O(30)$, $\alpha_3 \ll \alpha_2$ and the last two terms involving α_3 and α_4 can be neglected in the first perturbation solution.

Starting from the constitutive equation (24) and disregarding the normal-stress terms results in

$$S_{ij} = -P\delta_{ij} + 2\eta e_{ij} - 2\eta\tau de_{ij}/dt, \tag{25}$$

where $\tau = -\alpha_2/\alpha_1$ is the fluid time and $\eta = \alpha_1$ is the viscosity. For the first extreme case of low pressures and rigid roller surfaces, the fluid coefficients are assumed constant. For heavily loaded elasto-hydrodynamic lubrication, the viscosity is a function of the hydrostatic pressure. Equation (25), like the second-order equation, is the first perturbation to a Newtonian fluid for cases when the relaxation time τ is small compared with a characteristic time Δt of a periodic change in the flow near a fluid particle. The derivative d/dt considers only changes due to the linear velocity of the fluid particle; therefore Δt, in lubrication, is the transit time of a fluid particle along the bearing slit. Thus in the present study

$$N_{\mathrm{Deb}} = \tau/\Delta t = O(\tau U/l). \tag{26}$$

8. Differential equation of flow

Assuming rigid rollers for low pressures, the film thickness h in the neighbourhood of the minimum value h_m is (see figure 1)

$$h = h_m(1 + x^2/2Rh_m). \tag{27}$$

For gear-teeth or rollers with different radii of curvature, the equivalent radius is

$$R^{-1} = R_1^{-1} + R_2^{-1}. \tag{28}$$

The rollers are assumed infinitely long. As the lubricant is incompressible, the equation of continuity is

$$\partial u/\partial x + \partial v/\partial y = 0. \tag{29}$$

The film's thickness h $[= O(h_m)]$ is small compared with its length l $[= O(2Rh_m)^{\frac{1}{2}}]$. The continuity equation shows that $v/u = O(h_m^{\frac{1}{2}}/(2R)^{\frac{1}{2}})$. Under the usual assumptions of lubrication theory, terms of first and higher order in h_m/R are neglected throughout, relative to unity, as well as the inertia forces. The equilibrium equations are

$$S_{lm,m} = 0, \tag{30}$$

where $(\ )_{,i}$ denotes the partial derivative with respect to i. Substituting (25) into (30) yields the differential equation for steady-state flow,

$$P_{,x} = \eta u_{,yy} - \eta\tau(u_{,yx}u + u_{,yy}v)_{,y}, \tag{31}$$

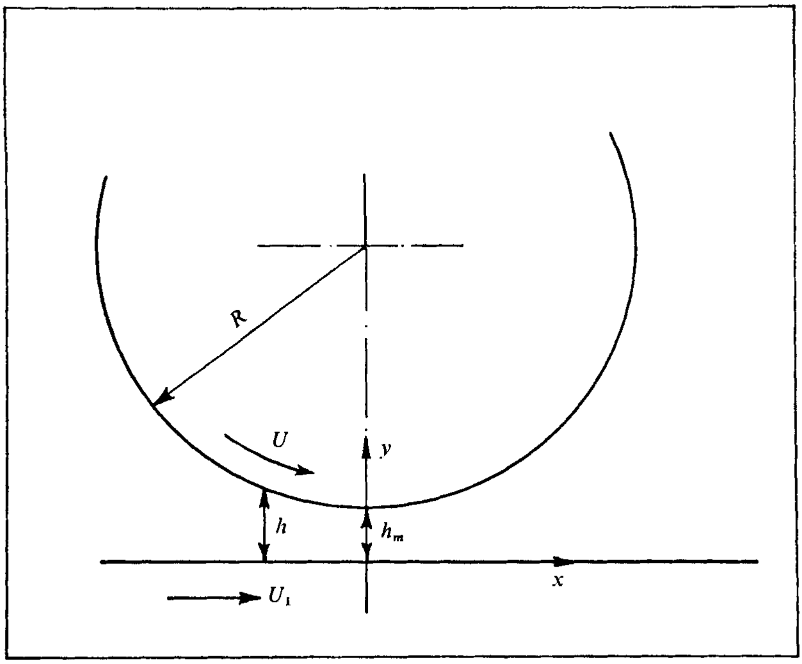

FIGURE 1. Hydrodynamics of cylinder and plane surface.

while $p_{,y}$ may be neglected [being $O(h_m/R)$]. Employing the dimensionless variables

$$\bar{u} = \frac{u}{U}, \quad \bar{v} = \frac{(2R)^{\frac{1}{2}}}{h_m^{\frac{1}{2}}} \frac{v}{U}, \quad \bar{x} = \frac{x}{(2Rh_m)^{\frac{1}{2}}}, \quad \bar{y} = \frac{y}{h_m} \tag{32}$$

and the ratio

$$\Gamma = \tau U/(2Rh_m)^{\frac{1}{2}}, \tag{33}$$

the flow equation becomes

$$\bar{u}_{,\bar{y}\bar{y}} - \Gamma(\bar{u}_{,\bar{y}\bar{x}}\bar{u} + \bar{u}_{,\bar{y}\bar{y}}\bar{v})_{,\bar{y}} = 2f(\bar{x}), \tag{34}$$

where

$$2f(\bar{x}) = \frac{h_m^{\frac{3}{2}}}{\eta U (2R)^{\frac{1}{2}}} P_{,\bar{x}}. \tag{35}$$

Γ is of the order of magnitude of the Deborah number N_{Deb}, i.e. the ratio of the relaxation time and the order of magnitude of the transit time of a fluid particle through the lubrication film. In the present investigation Γ and N_{Deb} are small compared with unity.

9. Velocity profile

To solve for small Γ a perturbation method is used, expanding in powers of Γ and retaining the first power only. We write

$$\bar{u} = \bar{u}_0 + \Gamma \bar{u}_1 + O(\Gamma^2), \tag{36}$$

$$\bar{v} = \bar{v}_0 + \Gamma \bar{v}_1 + O(\Gamma^2), \tag{37}$$

$$f(\bar{x}) = f_0(\bar{x}) + \Gamma f_1(\bar{x}) + O(\Gamma^2). \tag{38}$$

Substituting (36)–(38) into (34) and equating terms involving corresponding powers of Γ gives

$$\bar{u}_{0,\bar{y}\bar{y}} = 2f_0(\bar{x}), \tag{39}$$

$$\bar{u}_{1,\bar{y}\bar{y}} - (\bar{u}_{0,\bar{x}\bar{y}}\bar{u}_0 + \bar{u}_{0,\bar{y}\bar{y}}\bar{v}_0)_{,\bar{y}} = 2f_1(\bar{x}). \tag{40}$$

Denoting the ratio U_1/U of the circumferential velocities of the rollers by ϕ and denoting h/h_m by $\bar{h}$ the boundary conditions become

$$\bar{u}_0 = \phi, \quad \bar{u}_1 = 0 \quad \text{at} \quad \bar{y} = 0, \tag{41}$$

$$\bar{u}_0 = 1, \quad \bar{u}_1 = 0 \quad \text{at} \quad \bar{y} = \bar{h}. \tag{42}$$

Expanding the flux in a power series

$$q = q_0 + \Gamma q_1 + O(\Gamma^2 q) \tag{43}$$

and setting
$$h_i = 2q_i/h_m(U+U_1) \quad \text{for} \quad i = 0, 1, \tag{44}$$

the flux of the first and second velocity terms is

$$\int_0^{\bar{h}} \bar{u}_i d\bar{y} = \frac{q_i}{h_m U} = \frac{\bar{h}_i}{2}(1+\phi). \tag{45}$$

The solution of the velocity equation is

$$\bar{u} = \bar{u}_0 + \Gamma\bar{u}_1 = \alpha\bar{y}^4 + \beta\bar{y}^3 + \gamma\bar{y}^3 + \delta\bar{y}, \tag{46}$$

where

$$\alpha = \Gamma\bar{h}_{,\bar{x}}(1+\phi)^2\left(-\frac{9\bar{h}_e^2}{\bar{h}^7} + \frac{15\bar{h}_e}{\bar{h}^6} - \frac{6}{\bar{h}^5}\right), \tag{47}$$

$$\beta = \Gamma\bar{h}_{,\bar{x}}\left(\frac{18\bar{h}_e^2}{\bar{h}^6} - \frac{24\bar{h}_e}{\bar{h}^5} + \frac{8}{\bar{h}^4}\right) + \Gamma\bar{h}_{,\bar{x}}\phi\left(\frac{36\bar{h}_e^2}{\bar{h}^6} - \frac{60\bar{h}_e}{\bar{h}^5} + \frac{24}{\bar{h}^4}\right) + \Gamma\bar{h}_{,\bar{x}}\phi^2\left(\frac{18\bar{h}_e^2}{\bar{h}^6} - \frac{36\bar{h}_e}{\bar{h}^5} + \frac{16}{\bar{h}^4}\right),$$

$$\gamma = \frac{3}{\bar{h}^2}(1+\phi)\left(1-\frac{\bar{h}_e}{\bar{h}}\right) + \Gamma\bar{h}_{,\bar{x}}\left(-\frac{54}{5}\frac{\bar{h}_e^2}{\bar{h}^5} + \frac{9\bar{h}_e}{\bar{h}^4} - \frac{6}{5}\frac{1}{\bar{h}^3}\right)$$
$$+ \Gamma\bar{h}_{,\bar{x}}\phi\left(-\frac{108}{5}\frac{\bar{h}_e^2}{\bar{h}^5} + 36\frac{\bar{h}_e}{\bar{h}^4} - \frac{72}{5}\frac{1}{\bar{h}^3}\right) + \Gamma\bar{h}_{,\bar{x}}\phi^2\left(-\frac{54}{5}\frac{\bar{h}_e^2}{\bar{h}^5} + 27\frac{\bar{h}_e}{\bar{h}^4} - \frac{66}{5}\frac{1}{\bar{h}^3}\right), \tag{49}$$

$$\delta = (1+\phi)(3\bar{h}_e/\bar{h}^2 - 2/\bar{h}) + \Gamma(\tfrac{4}{5}\bar{h}^3\alpha + \tfrac{1}{2}\bar{h}^2\beta). \tag{50}$$

10. Bearing force

Integrating (31) with respect to x and using the velocity solution (46)–(50), we have

$$P = 6(1+\phi)\eta U\left(\frac{2R}{h_m^3}\right)^{\frac{1}{2}}\int_{-\infty}^{\bar{x}}\left(\frac{1}{\bar{h}^2} - \frac{\bar{h}_e}{\bar{h}^3}\right)d\bar{x}$$
$$+ \Gamma\eta U\left(\frac{2R}{h_m^3}\right)^{\frac{1}{2}}\left[\frac{9}{10}\frac{\bar{h}_e}{\bar{h}^4}(1+\phi)^2 + \frac{2}{5}\frac{1}{\bar{h}^2}(-2+\phi-2\phi^2)\right] + K. \tag{51}$$

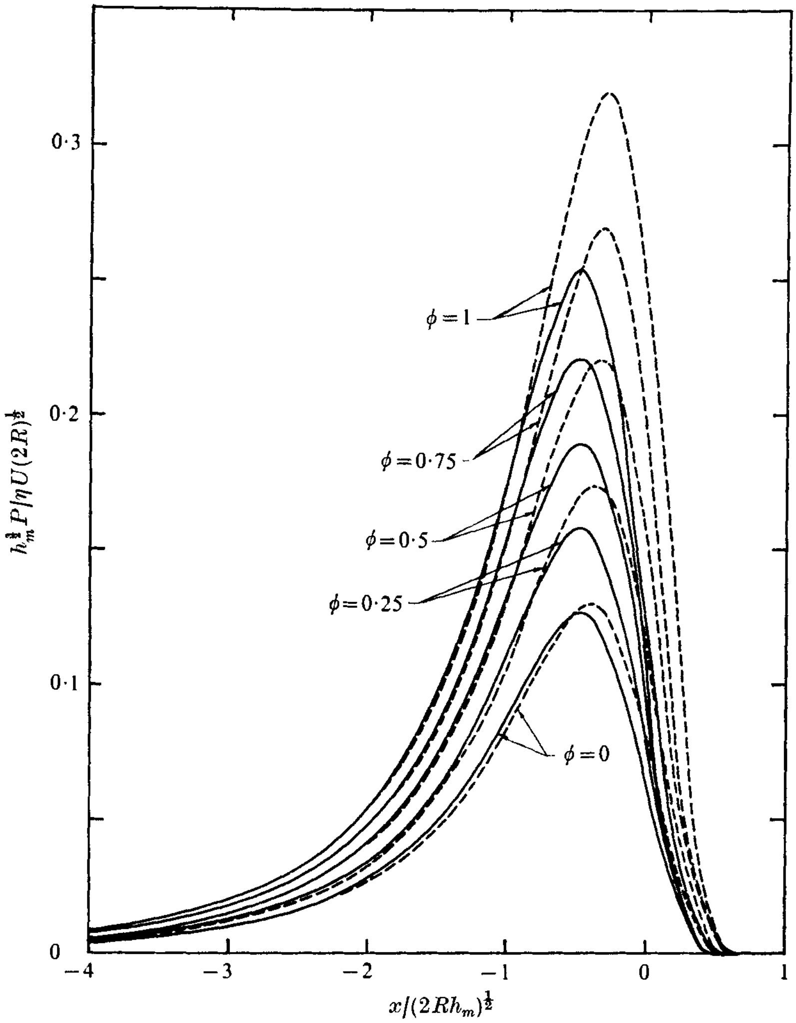

FIGURE 2. Pressure profiles for Newtonian and elastico-viscous lubricants. – – –, $\Gamma = 0{\cdot}2$; ——, $\Gamma = 0$.

When $\phi = 0$ pure sliding occurs, while for $\phi = 1$ there is only rolling. The force W is given by

$$W = L\int_{-\infty}^{x_f} P\,dx, \tag{52}$$

where L is the roller length and x_f corresponds to the end of the pressure zone.

The usual boundary conditions for the pressure wave are

$$P = 0 \quad \text{at} \quad x = -\infty \tag{53}$$

and

$$P = \partial P/\partial x = 0 \quad \text{at} \quad x = x_f. \tag{54), (55}$$

The first condition (53) shows that $K = 0$ in (51). From the two additional conditions (54) and (55) the flux constant $\bar{h}_e$ and x_f have been obtained by numerical iteration.

Figure 2 shows the pressure distributions for both Newtonian ($\Gamma = 0$) and

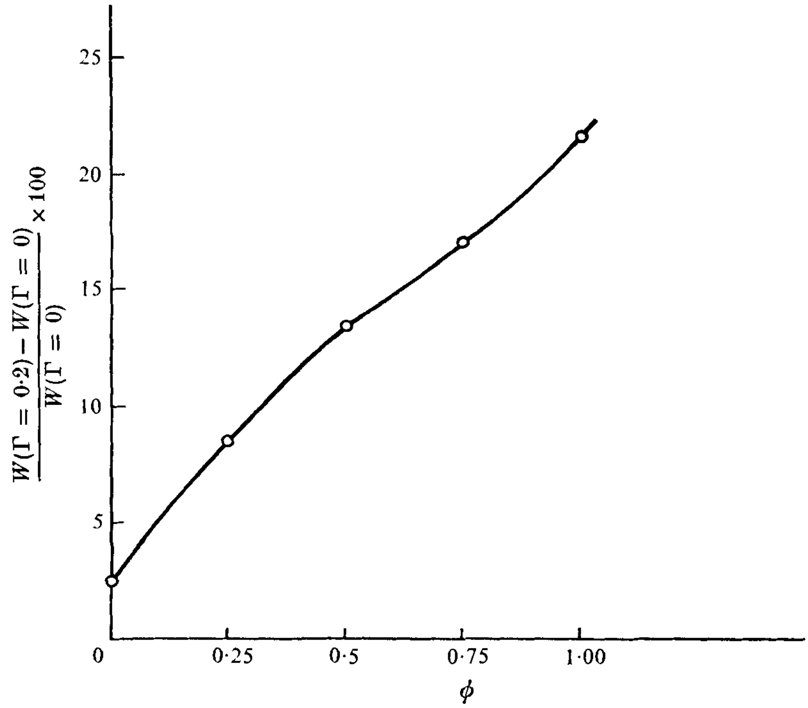

FIGURE 3. Relative improvement in load capacity.

elastico-viscous lubricant ($\Gamma = 0\cdot2$). The results are presented for different values of rolling and sliding, represented by the ratio ϕ. The figure demonstrates how elasticity of the fluid increases the pressures and the load capacity. The relative improvement in the load capacity increases with ϕ, i.e. the role of elasticity of the fluid is more effective in rolling than in sliding (see figure 3).

11. Elastohydrodynamic lubrication

In this section the simplifying assumptions of Ertel–Grubin (see Cameron 1966) are adopted, and their theory is extended to elastico-viscous lubricants. They assumed that the Hertzian shape of surfaces outside the Hertzian contact zone is the same as without lubricant, while the film thickness is constant in the contact region.

The width of the Hertz contact region is $2a$, where

$$a = \left(\frac{4}{E_r}\frac{WR}{L}\right)^{\frac{1}{2}} \tag{56}$$

and

$$\frac{1}{E_r} = \frac{1-\sigma_1^2}{\pi E_1} + \frac{1-\sigma_2^2}{\pi E_2}, \tag{57}$$

E and σ being Young's modulus and Poisson's ratio respectively. For $|x| < a$ the gap width h_m is constant and for $|x| > a$

$$h = h_m + h_s, \tag{58}$$

where
$$h_s = \frac{2W}{LE_r}\left\{\frac{x}{a}\left(\frac{x^2}{a^2}-1\right)^{\frac{1}{2}} - \ln\left[\frac{x}{a}+\left(\frac{x^2}{a^2}-1\right)^{\frac{1}{2}}\right]\right\}. \tag{59}$$

The variation of the viscosity with pressure is

$$\eta_p = \eta_0 e^{\alpha P}, \tag{60}$$

η_0 being the atmospheric pressure viscosity. The solution for the pressure distribution is similar to (51) but instead of P we have to write P_0, where

$$P_0 = (1-e^{-\alpha P})/\alpha. \tag{61}$$

In heavily loaded rollers αP is large enough that P_0 in the contact zone can be taken as constant and equal to $1/\alpha$ (hence $\partial P_0/\partial x = 0$). We employ the non-dimensional film thickness H and pressure P_0^* defined by

$$H = \frac{LE_r}{W}h, \quad P_0^* = \frac{(W/LE_r)^2}{6U\eta_0 a}P_0. \tag{62), (63}$$

For pure rolling ($\phi = 1$), P_0^* in the contact zone is

$$P_0^* = 2\int_{-\infty}^{-1}\frac{H_s}{(H_m+H_s)^3}+0{\cdot}4\Gamma\frac{1}{H_m^2}. \tag{64}$$

Substituting the value of the integral according to Ertel–Grubin we get

$$P_0^* = 0{\cdot}1972H_m^{-\frac{11}{8}}+0{\cdot}4\Gamma H_m^{-2}. \tag{65}$$

From (63) and (56) we have (for $P_0 = 1/\alpha$)

$$P_0^* = \frac{(W/LEr)^2}{6U\eta_0}\left(\frac{E_r L}{4WR}\right)^{\frac{1}{2}}\frac{1}{\alpha}. \tag{66}$$

The relation between the non-dimensional minimum film thickness

$$H_m = (LE_r/W)\,h_m$$

and P_0^* is shown in figure 4. h_m is appreciably larger than the Newtonian gap width for highly loaded rollers with small values of H_m.

12. Conclusions

For the two extreme cases of low and high loads, an increase in the load capacity compared with the Newtonian value is obtained. This means an increase in film thickness for the same external load. This improvement in lubrication is more pronounced for rolling than for sliding and increases rapidly as the film thickness decreases, thereby possibly preventing surface failure.

Appendix

Rotation invariance

The following proof shows that the same derivative d/dt (with respect to the material) is obtained whether **e** is in static or relatively rotating co-ordinates. It is possible to compare the components of the derivatives at the time when the

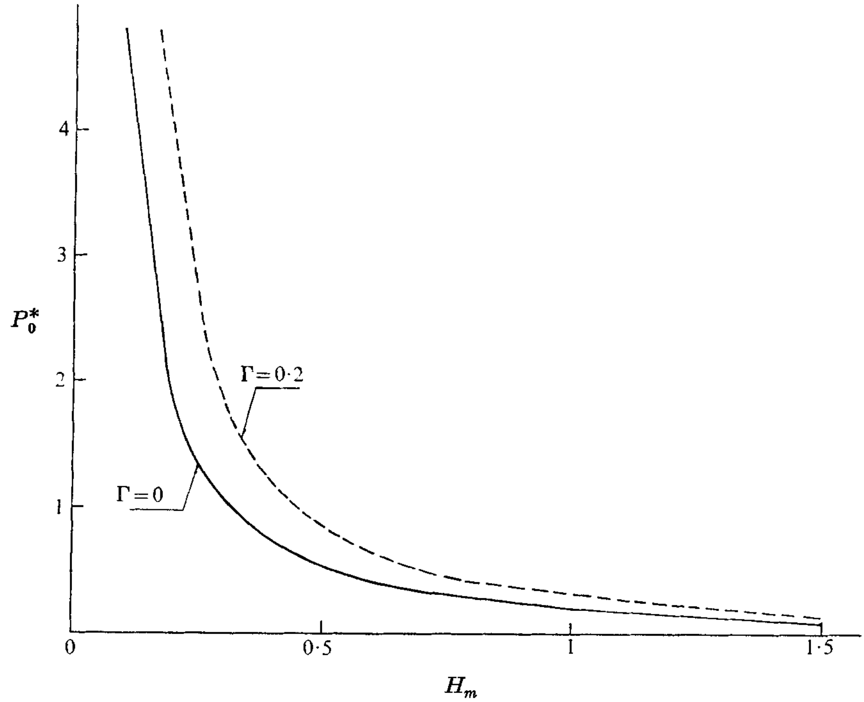

FIGURE 4. P_0^* *vs.* H_m for Newtonian and elastico-viscous lubricants.

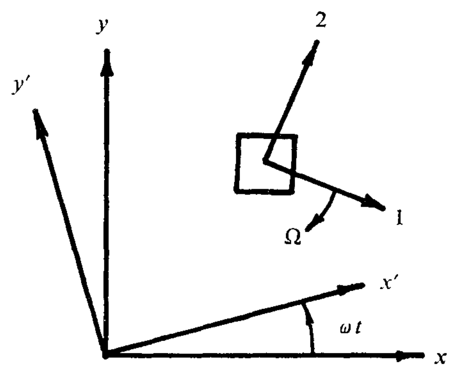

FIGURE 5. Static and rotating co-ordinates.

static and rotating systems coincide. x_i is a static rigid orthogonal co-ordinate system, while x_i' is rotating with angular velocity components ω_{ij}; see figure 5. The rate of-strain-tensor has principal co-ordinates (1, 2, 3) rotating with angular velocity components Ω_{ij} relative to x_i. The direction of rotation of x_i' is counter-clockwise, but the direction of the rotation $\mathbf{\Omega}$ of (1, 2, 3) is clockwise [like the rotation of the fluid particle in (4)].

Defining

$$\dot{\mathbf{e}} \equiv \frac{\partial \mathbf{e}}{\partial t} + \frac{\partial \mathbf{e}}{\partial x_\alpha} v_\alpha, \tag{A 1}$$

(21) becomes

$$d\mathbf{e}/dt = \dot{\mathbf{e}} - \mathbf{\Omega}\mathbf{e} + \mathbf{e}\mathbf{\Omega}. \tag{A 2}$$

Transformation to rotating co-ordinates

According to the definition (A 2), the last two rotation terms must accompany d/dt also in x_i' co-ordinates. But instead of $\mathbf{\Omega}$ one has to substitute the relative angular velocity $\mathbf{\Omega}+\boldsymbol{\omega}$ of the principal co-ordinates $(1, 2, 3)$ and x_i'. The time derivative in x_i' is

$$d\mathbf{e}'/dt = \dot{\mathbf{e}}' - (\mathbf{\Omega}+\boldsymbol{\omega})\,\mathbf{e}' + \mathbf{e}'(\mathbf{\Omega}+\boldsymbol{\omega}). \tag{A 3}$$

The rule of transformation is

$$\mathbf{e}' = \mathbf{ReR}^*. \tag{A 4}$$

Differentiating with respect to time results in

$$\dot{\mathbf{e}}' = \mathbf{R}\dot{\mathbf{e}}\mathbf{R}^* + \dot{\mathbf{R}}\mathbf{e}\mathbf{R}^* + \mathbf{R}\mathbf{e}\dot{\mathbf{R}}^*. \tag{A 5}$$

At the time when x_i and x_i' coincide (assuming at $t = 0$) we have

$$\mathbf{R} = \mathbf{I}, \quad \mathbf{R}^* = \mathbf{I}, \tag{A 6}$$

$$\dot{\mathbf{R}} = \boldsymbol{\omega} \quad \dot{\mathbf{R}}^* = \boldsymbol{\omega}^*. \tag{A 7}$$

Substitution in (A 5) yields

$$\dot{\mathbf{e}}'(t=0) = \dot{\mathbf{e}} + \boldsymbol{\omega}\mathbf{e} - \mathbf{e}\boldsymbol{\omega}. \tag{A 8}$$

Substituting (A 8) in (A 3) results in identical terms to (A 2). So d/dt is independent of $\boldsymbol{\omega}$ and is thus rotation invariant.

It is advantageous to show detailed transformation of the two-dimensional tensor $\mathbf{e}$. The x_i' co-ordinate system and $(1, 2, 3)$ are rotating around the axis $z \equiv z'$ with angular velocities ω and Ω respectively, in the directions shown in figure 5. Thus

$$\mathbf{e} = \begin{bmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{bmatrix}, \tag{A 9}$$

$$\mathbf{R} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}. \tag{A 10}$$

Performing the transformation and differentiating with respect to time results, for $t = 0$, in

$$\dot{\mathbf{e}}'(t=0) = \begin{bmatrix} \dot{e}_{11} + 2\omega e_{12} & \dot{e}_{12} - \omega(e_{11} - e_{22}) \\ \dot{e}_{12} - \omega(e_{11} - e_{22}) & \dot{e}_{22} - 2\omega e_{12} \end{bmatrix}. \tag{A 11}$$

The additional rotational terms are

$$-(\mathbf{\Omega}+\boldsymbol{\omega})\,\mathbf{e} + \mathbf{e}(\mathbf{\Omega}+\boldsymbol{\omega}) = (\Omega+\omega)\begin{bmatrix} -2e_{12} & (e_{11}-e_{22}) \\ (e_{11}-e_{22}) & 2e_{12} \end{bmatrix}. \tag{A 12}$$

Adding (A 12) to (A 11) results in

$$\frac{d}{dt}e_{ij}'(t=0) = \frac{d}{dt}\mathbf{e}'(t=0) = \begin{bmatrix} \dot{e}_{11} + 2\Omega e_{12} & \dot{e}_{12} - \Omega(e_{11} - e_{22}) \\ \dot{e}_{12} - \Omega(e_{11} - e_{22}) & e_{22} - 2\Omega e_{12} \end{bmatrix}. \tag{A 13}$$

This result is independent of ω.

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